

Spatial three-player prisoners' dilemma

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This paper extends traditional two-player prisoners' dilemma (PD) to three-player PD. We have studied spatial patterns of cooperation behaviors, growth patterns of cooperator clusters and defector clusters, and cooperation frequency of the players. It is found while three-player PD exhibits many properties similar to two-player PD, some new features arise. Specifically, (i) a new region appears, in which neither a 3×3 cooperator cluster nor a 3×3 defector cluster could grow; (ii) more growth patterns of cooperator clusters and defector clusters are identified; (iii) multiple cooperation frequencies exist in the region that exhibits dynamic chaos. Some theoretical analysis of these features is presented.

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I. INTRODUCTION

Game theory is a unifying paradigm behind many scientific disciplines, from biology to behavioral sciences to economics. The widely studied games include the Hawk-Dove game [1–3], prisoners' dilemma [4–8], public good game [9,10], Snowdrift game [11–13], rock-scissors-paper game [14], and so on (see, e.g., a recent review [15], and references therein).

The prisoners' dilemma (PD) has been studied as a metaphor for problems surrounding the evolution of cooperative behavior. In the standard form, it is played by two players, each of whom could choose either to cooperate C or to defect D in any one encounter. If both players choose C , both get a payoff of magnitude R ; if one defects while the other coop-

erates, D gets the game's biggest payoff T , while C gets S ; if both defect, both get P . The payoff matrix is therefore as follows:

$$\begin{array}{c|cc} & C & D \\ \hline C & R & S \\ \hline D & T & P \end{array} \tag{1}$$

With $T > R > P > S$, the paradox is evident (We would like to mention that usually P and S are set to zero, R and T are set to $R=1, 1 < T=b < 2$). In single encounter D is more advantageous but for the whole unity, C is more beneficial. When it is played in a single round, defectors have more fitness over cooperators. But when the game is repeated and coop-

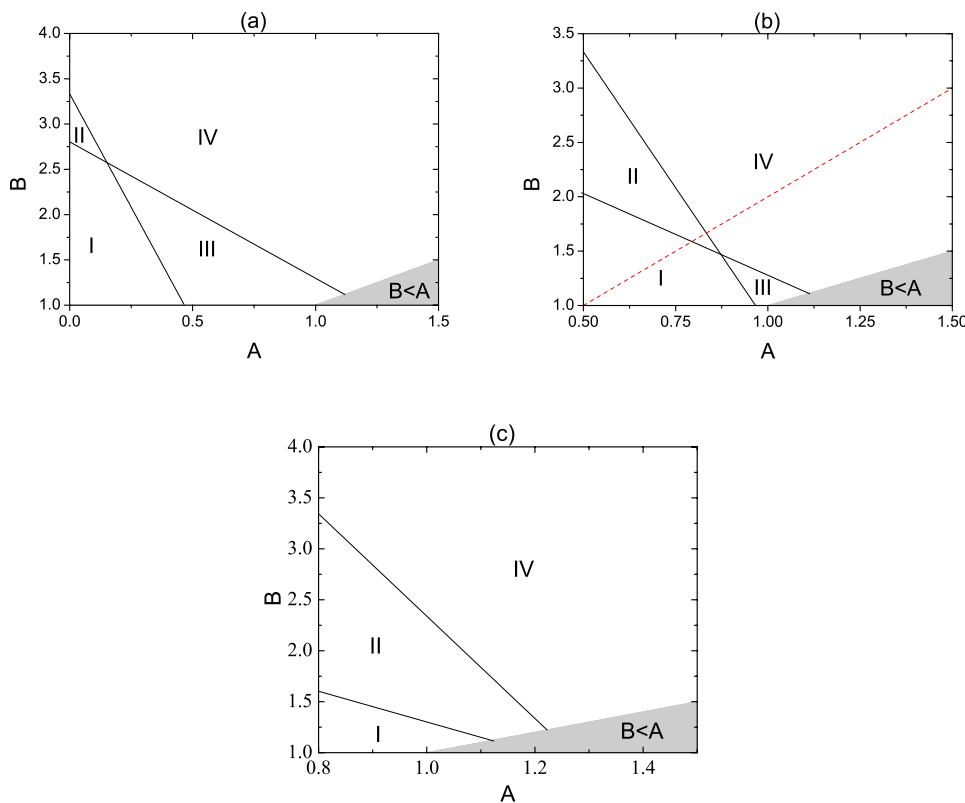


FIG. 1. (Color online) Four regions in the space (A, B) . The gray region corresponds to $B < A$ and is not studied as explained in main text. (a) $p=0$; (b) $p=1/2$, the red dashed line represents $B=2A$; (c) $p=0.8$, note that region III disappears.

erators meet often, there is possibility that cooperators might have better fitness than defectors.

The two-player PD has been widely studied. In their seminal work, Nowak and May introduced a spatial evolutionary game to demonstrate that local interactions within a spatial structure can maintain cooperative behavior indefinitely and lead to spatial dynamic fractals [6,7]. Szabó *et al.* have investigated evolutionary PD with voluntary participation [16]. Recently, PD on social networks and evolving networks has also been discussed [17].

There are many socially and economically important examples where the number of decision makers involved is greater than 2, see, e.g., Refs. [18,19]. In terms of PD, the situation usually is modeled as repeated play of simple pair interactions, in which payoffs are calculated as the average of the payoffs against the two opponents in the two-player PD. For example, the two-player PD has been generalized to three-player PD in Refs. [20,21]. Specifically, the payoff matrix is

$$\begin{array}{c|ccc} & CC & CD & DD \\ \hline C & 1 & 1/2 & 0 \\ D & b & b/2 & 0 \end{array} \quad (2)$$

In this three-player PD, there is also only one parameter b . As shown in Sec. IV B, the three-player PD is qualitatively similar to classical two-player PD.

In this paper, we study a different three-player PD on spatial 2D lattice, in which we consider the effect of three parameters. It is found qualitatively different results arise. Some theoretical analysis of these results is presented.

The paper is organized as follows. In the next section, three-player PD model is presented. We study the replicator dynamic of three-player PD in well-mixed population in Sec. III. The simulation and analytical investigations are reported in Sec. IV. The conclusion is given in Sec. V.

II. MODEL

In our three-player PD model, the payoff matrix of three player PD is

$$\begin{array}{c|ccc} & CC & CD & DD \\ \hline C & R & p & S \\ D & T & r & P \end{array} \quad (3)$$

It is clear that T is the largest among the six parameters while S is the smallest [22]. In a three-player PD game, it is reasonable to assume $R > p$, $r > P$ (the payoffs increase with an increasing number of cooperators), and $r > p$ (defectors always receive higher payoffs than cooperators). Another reasonable assumption is $R > P$ [23].

In our model, we set $P=S=0$, $R=1$, $r=A$, $T=B$. Here $B > 1$, $B > A > p \geq 0$, $p < 1$. As a result, we have three parameters A, B, p . In next section, we study the effect of B and A

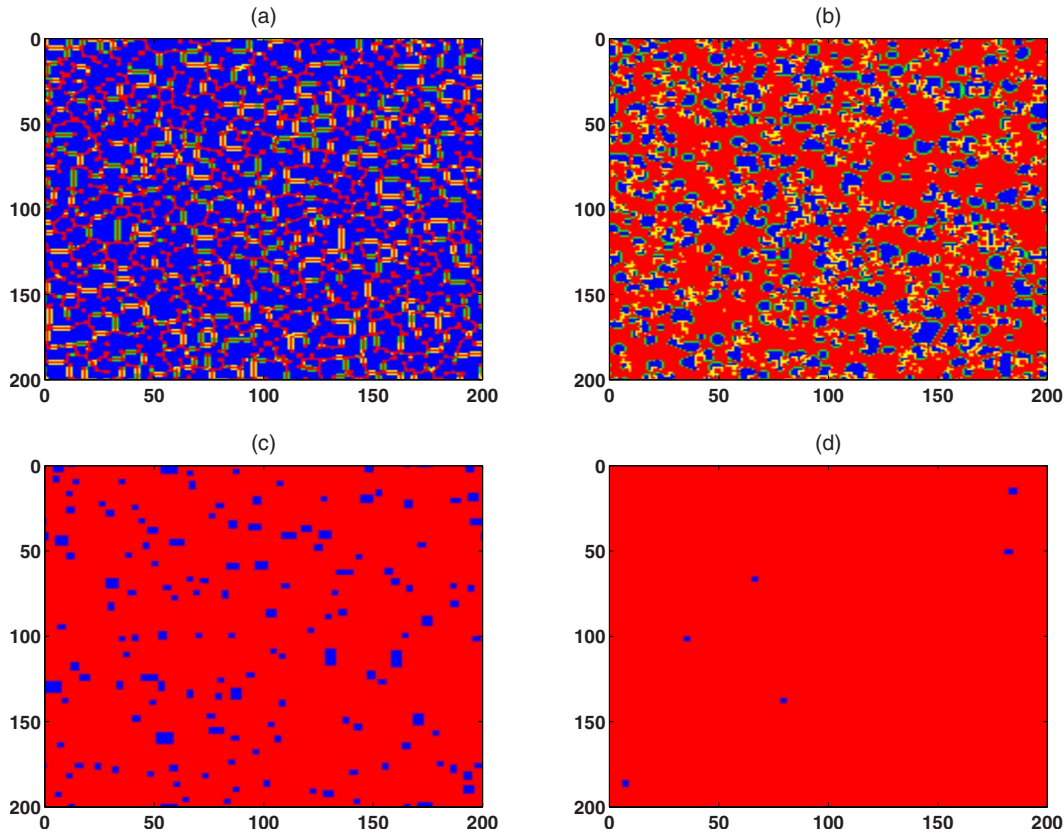


FIG. 2. (Color online) Typical spatial patterns in region (a) I; (b) II; (c),(d) IV. $p=0$, (a) $B=2.0$, $A=0.1$; (b) $B=3.0$, $A=0.02$; (c) $B=3.0$, $A=0.16$; (d) $B=5.0$, $A=0.16$. Initially 10% defectors are randomly distributed.

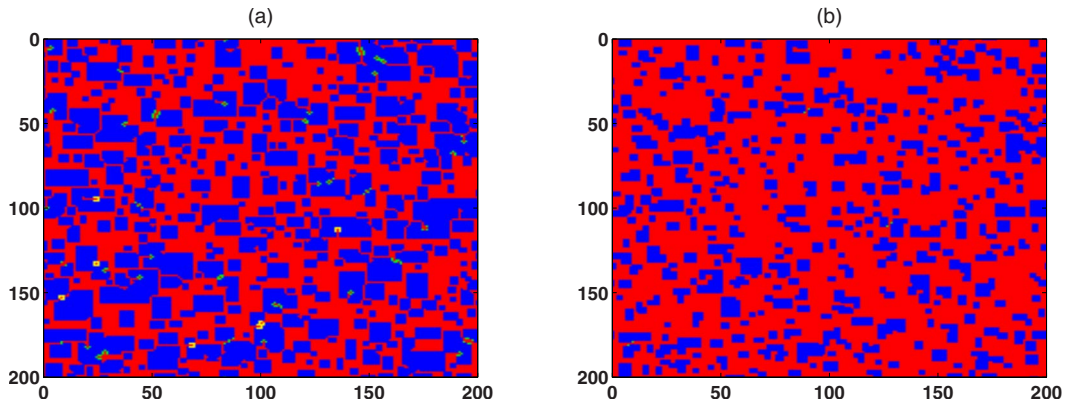


FIG. 3. (Color online) Typical spatial patterns in region III. $p=0$, (a) $B=1.37, A=0.4$; (b) $B=2.0, A=0.4$. Initially 10% defectors are randomly distributed.

on the cooperation behavior under different values of p . Note that in the special case $p=1/2, B=2A$, the three-player PD reduces to those corresponding to Eq. (2).

III. REPLICATOR DYNAMIC

We first investigate the replicator dynamic of three-player PD in well-mixed population. Note that this analysis is valid only in the limit of an infinite population. Denote the frequency of cooperator as x , then the fitness of C is

$$f_C = x^2 + p \times 2x(1-x).$$

The fitness of D is

$$f_D = Bx^2 + A \times 2x(1-x).$$

The average fitness is

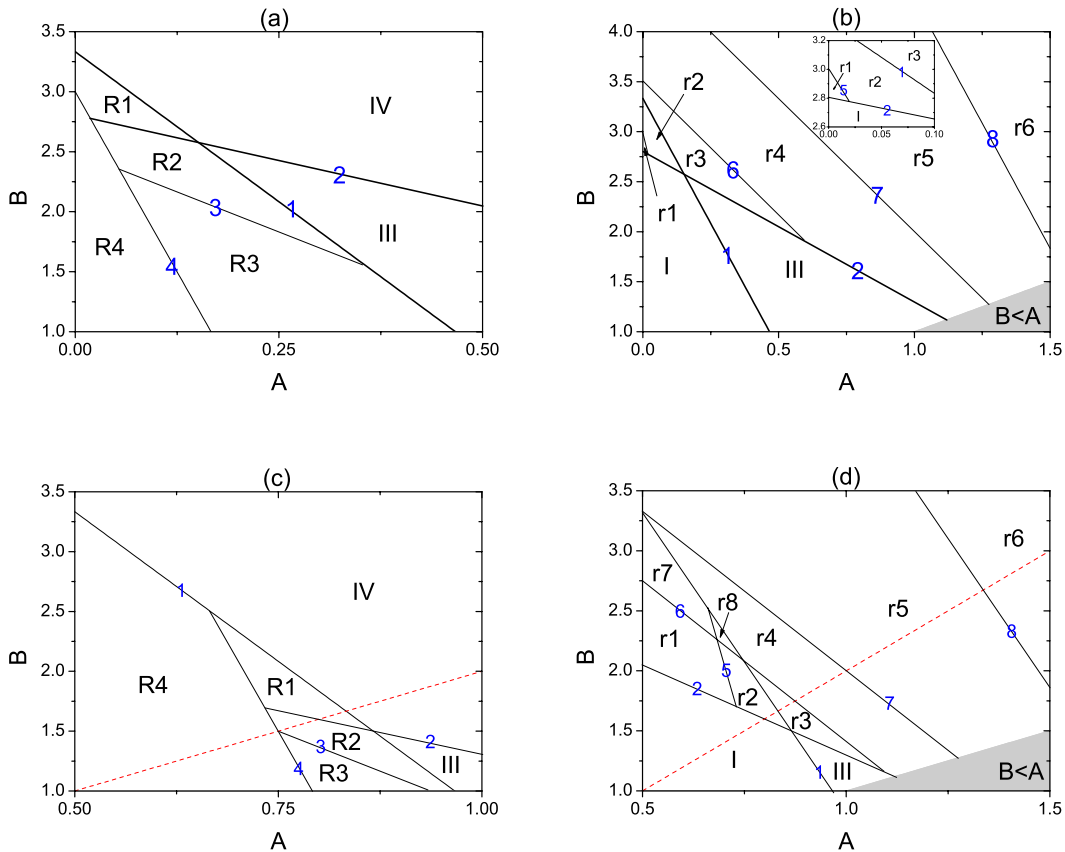


FIG. 4. (Color online) (a),(c) Regions R1-R4 correspond to different growth pattern of a 3×3 C cluster. (b),(d) Regions r1-r8 correspond to different growth pattern of a 3×3 D cluster. The inset in (b) shows details of region r1. (a),(b) $p=0$; (c),(d) $p=1/2$. The red dashed line represents $B=2A$. The blue numbers on boundaries denote the number of boundaries (see Table I).

$$\begin{aligned}\phi &= xf_C + (1-x)f_D \\ &= x^3 + 2px^2(1-x) + (1-x)[Bx^2 + 2Ax(1-x)].\end{aligned}$$

The replicator equation is given by

$$\begin{aligned}\dot{x} &= x(f_C - \phi) \\ &= x\{x^2 + p \times 2x(1-x) - 2px^2(1-x) \\ &\quad - x^3 - (1-x)[Bx^2 + 2Ax(1-x)]\} \\ &= x^2(x-1)[(B-2A-1+2p)x + 2(A-p)].\end{aligned}\quad (4)$$

Setting $\dot{x}=0$, we can obtain that

$$x=0 \text{ or } x=1 \text{ or } x = \frac{2(A-p)}{2A+1-B-2p}.$$

Since $B>1$ and $A>p$, we have $2(A-p)>2A+1-B-2p$. This means that $\frac{2(A-p)}{2A+1-B-2p}>1$ (when $2A+1-B-2p>0$) or $\frac{2(A-p)}{2A+1-B-2p}<0$ (when $2A+1-B-2p<0$). As a result, the solution $x = \frac{2(A-p)}{2A+1-B-2p}$ should be discarded.

Consequently, Eq. (4) has two fixed points: at $x=1$ there is an unstable equilibrium where everybody cooperates; at $x=0$ there is a stable equilibrium where everybody defects. Therefore $x=0$ is the global attractor of these dynamics. Hence, three-player PD predicts the victory of defectors in well-mixed populations as in two-player PD.

Now let us discuss a more general case $R=p=S=0$, that is, we allow $P>S$. In this case, we have

$$f_C=0$$

and

$$f_D = Bx^2 + 2Ax(1-x) + P(1-x)^2.$$

Thus

$$\phi = xf_C + (1-x)f_D = (1-x)[Bx^2 + 2Ax(1-x) + P(1-x)^2]$$

and

$$\dot{x} = x(f_C - \phi) = -x(1-x)[Bx^2 + 2Ax(1-x) + P(1-x)^2].$$

Suppose there exists one internal fixed point, i.e., $0<x=x_1<1$ enables $\dot{x}=0$, then we have

$$Bx_1^2 + 2Ax_1(1-x_1) + P(1-x_1)^2 = 0,$$

which requires either

$$B=A=P=0$$

or at least one of the three parameters B, A, P is negative. This contradicts with the assumption of PD, i.e., defectors always receive higher payoffs than cooperators ($B>R=0, A>p=0, P>S=0$).

Therefore, in the line of PD, there is no internal fixed point in the general case $R=p=S=0$, either. Thus in the remainder we will restrict to the case $P=S$, as often done for the two-player case.

IV. RESULTS

In this section, we study the spatial three-player PD on 2D lattice under different values of p . First we consider the case $p=0$.

A. $p=0$

We study the spatial three-player PD, in which the game is played in a 200×200 array. We assume that in each generation, a player could play the game with his/her eight

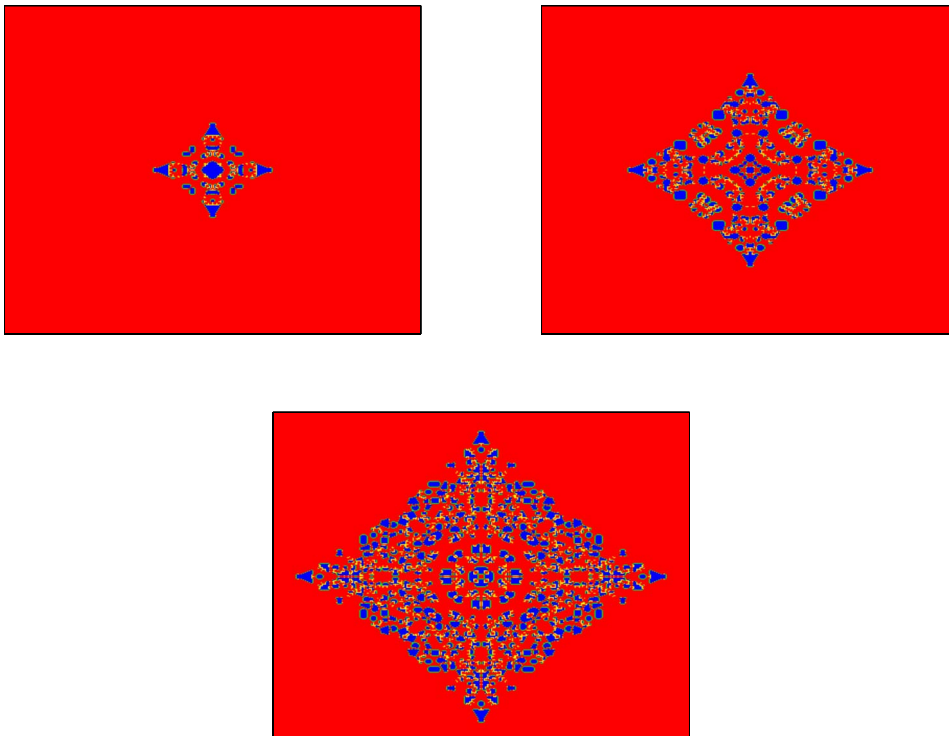


FIG. 5. (Color online) Typical growth patterns in region R1. The patterns correspond to $t=60, 120, 180$.

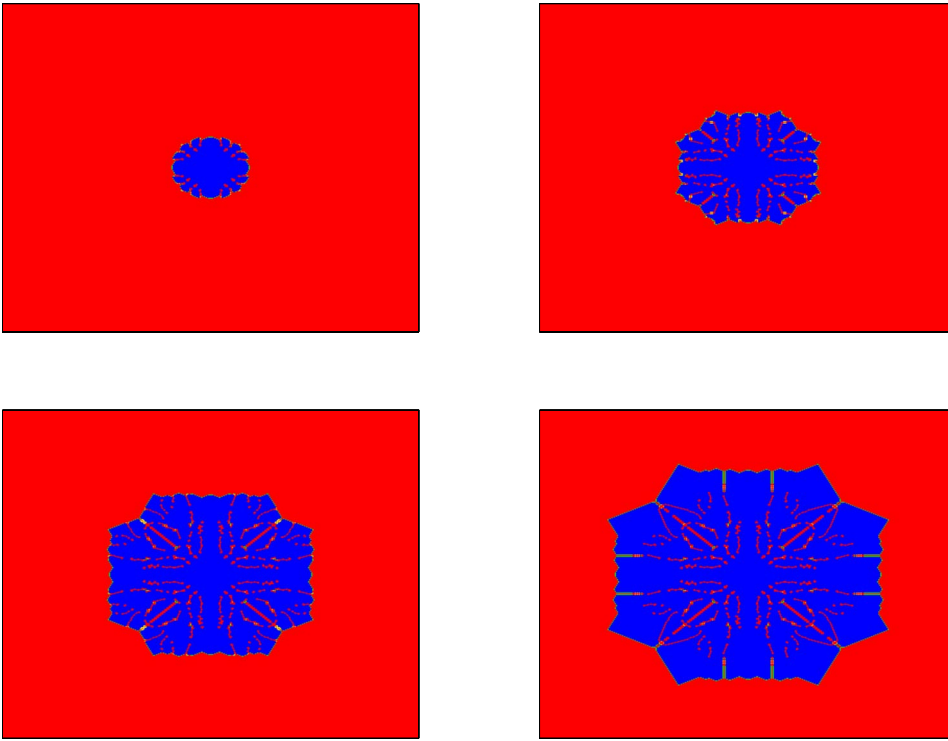
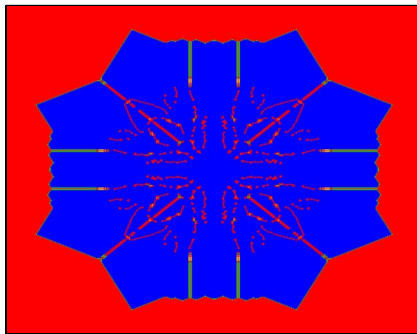


FIG. 6. (Color online) Typical growth patterns in region R2. The patterns correspond to $t = 60, 120, 180, 240, 300$.



neighbors. As the game involves three players in a round, we choose to have all the possible combinations of two neighbors to play. So a player will play $C_8^2=28$ times at one generation. The score for him is the sum of the payoffs in these 28 encounters. Then at the start of the next generation, each lattice site is occupied by the player with the highest score among the previous owner and the eight immediate neighbors. If the highest score is achieved at several sites, then the lattice site will be occupied by a random selected

site from those with the highest scores. Note the site does not stay preferably with its strategy when the site itself is among those with highest payoff. In the results shown below, periodic boundary conditions are used. We have also tested fixed boundary condition, in which the players on the boundary have less neighbors and play less times, and no qualitative change is found.

Figure 1 shows in the space (A, B) , four regions are classified. A 3×3 cluster of C will grow in regions I and II. In

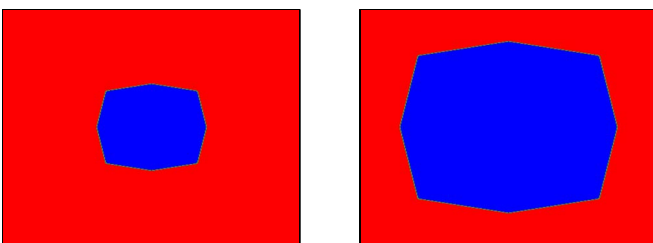


FIG. 7. (Color online) Typical growth patterns in region R3. The patterns correspond to $t=90, 180$.

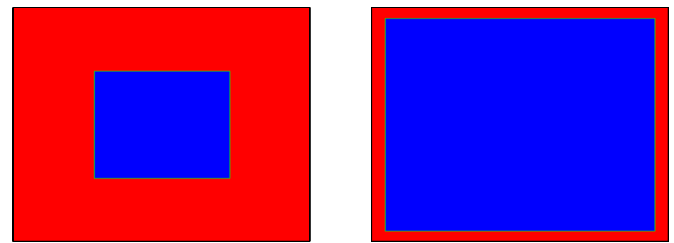
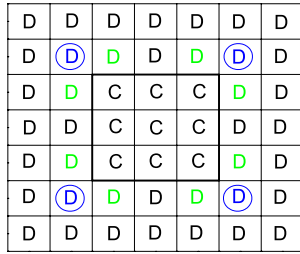
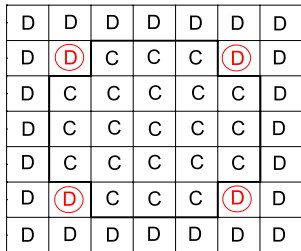


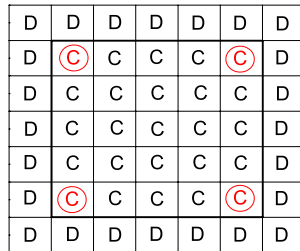
FIG. 8. (Color online) Typical growth patterns in region R4. The patterns correspond to $t=90, 180$.



(a)



(b)



(c)

FIG. 9. (Color online) (a) Pattern of a 3×3 cluster of C. (b),(c) patterns after one generation. (b) shows the pattern in regions R1-R3; (c) shows the pattern in region R4.

contrast, it will not grow in regions III and IV. Furthermore, a 3×3 cluster of D will grow in regions II and IV and it will not grow in regions I and III.

Figure 2 shows typical spatial patterns in regions I, II, IV. The color coding is the same as in Ref. [6]: blue represents a C site that was C in the preceding generation; red, a D site following a D; yellow, D following a C; and green, C following a D. Figure 2(a) shows the typical irregular and relatively static network of “D lines” against a background of C in region I. Figure 2(b) shows the typical pattern of the

TABLE I. The analytical expressions of boundaries (cf. Figs. 4, 19, and 20).

boundary 1	$3B+15A=10+15p$
boundary 2	$10B+15A=28$
boundary 3	$6B+16A=15+12p$
boundary 4	$B+12A=3+15p$
boundary 5	$B+12A=3+15p$
boundary 6	$6B+16A=21+7p$
boundary 7	$6B+16A=28$
boundary 8	$3B+15A=28$

dynamic chaos in region II. Figures 2(c) and 2(d) show the typical patterns in region IV, in which most sites are occupied by D and existing C form a stable cluster. With the increase of B and/or A, the number of stable C clusters decreases. When B and/or A are large, all players become defectors.

In the spatial two-player PD, the payoffs of the matrix are usually set to $R=1$, $T=b>1$, $S=P=0$. As a result, static network of “D lines” against a background of C is observed when $b<8/5$, and dynamic chaos are observed when $8/5<b<5/3$. When $b>5/3$, most sites are occupied by D and existing C form stable cluster. Therefore, regions I, II, IV in three player PD are similar to regions $b<8/5$, $8/5<b<5/3$, and $b>5/3$, respectively. Note here we do not consider self-interaction in spatial two-player PD in order for consistency with three-player PD (see Ref. [7]).

Next we focus on region III. Figure 3 shows typical patterns in region III. One can see stable C clusters exist. With the increase of B and/or A, the number of stable C clusters

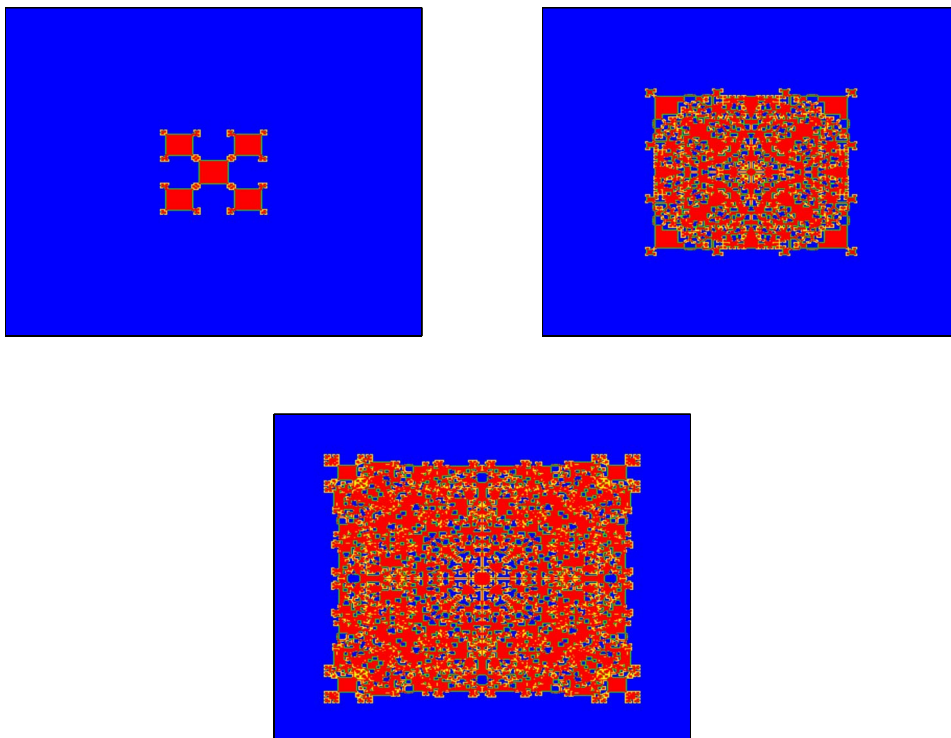


FIG. 10. (Color online) Typical growth patterns in region I. The patterns correspond to $t=50, 100, 150$.

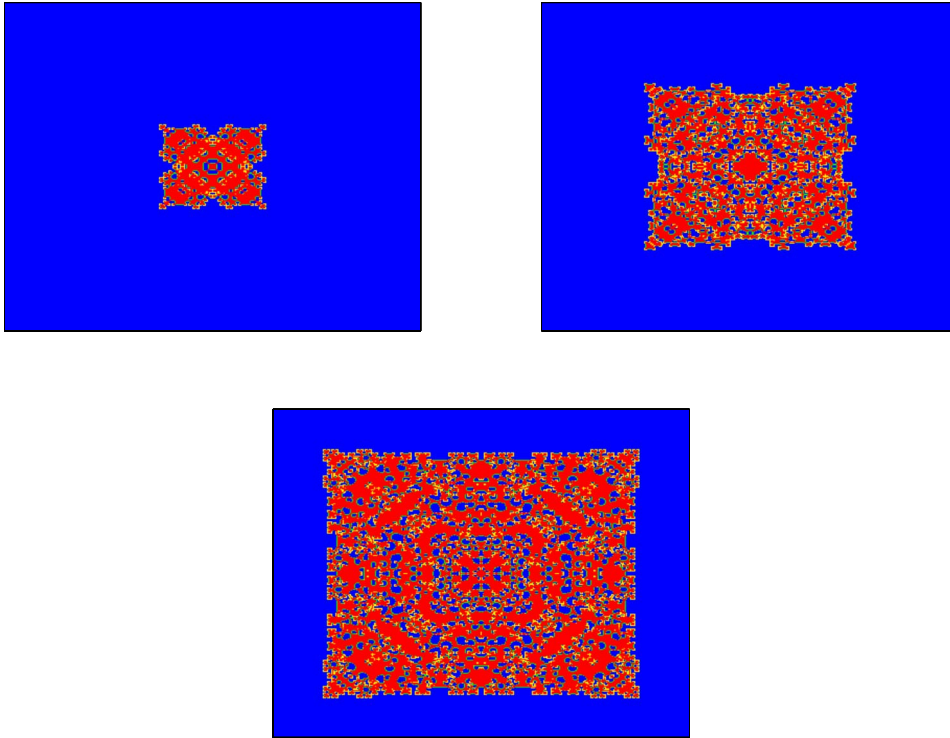


FIG. 11. (Color online) Typical growth patterns in region r2. The patterns correspond to $t=50, 100, 150$.

decreases. Although quantitatively patterns in region III have more C clusters than that in region IV, the patterns are qualitatively similar to each other in these two regions. This is because in region III, (i) a single D can grow into a 3×3 D cluster after one generation; (ii) 3×3 D clusters will not

grow if they are well separated from each other. But D clusters that are not well separated could grow. As a result, randomly distributed defectors have great chance to grow and merge together to form large D clusters. For example, assume there are four defectors surrounded by cooperators, and their locations are (i, j) , $(i, j+5)$, $(i+5, j)$, $(i+5, j+5)$. It is easy to find out these four defectors could grow into a large D cluster consisting of 64 defectors.

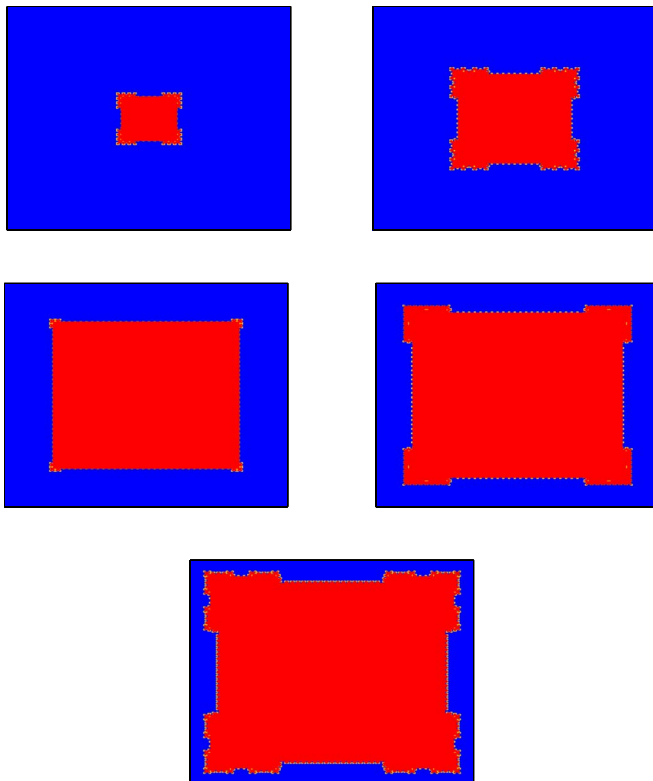


FIG. 12. (Color online) Typical growth patterns in region r3. The patterns correspond to $t=45, 90, 135, 160, 180$.

We study growth pattern of a 3×3 cluster of C in regions I and II. It is found there are four kinds of growth patterns, corresponding to the four regions shown in Fig. 4(a). The typical growth patterns are shown in Figs. 5–8.

We investigate the boundaries between the regions. Figure 9(a) shows the pattern of a 3×3 cluster of C . Figures 9(b) and 9(c) shows patterns after one generation. Figure 9(b) shows the pattern in regions R1-R3 and Fig. 9(c) shows the pattern in region R4. Difference between the patterns is shown by sites with red colored circled C or D . These sites are originally in the D state, and they have seven D neighbors and one C neighbor (see sites with blue colored circled D). The C neighbor has three C neighbors, and its payoff $P_C=3$. For the seven D neighbors, it is obvious the maximum payoff is achieved on sites with green colored D . These

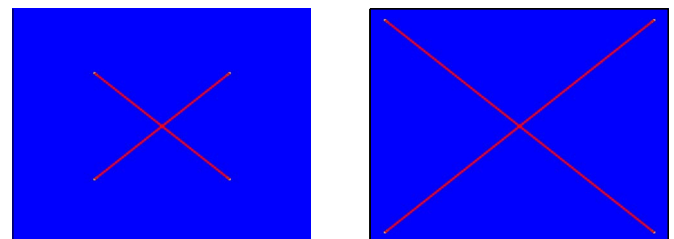


FIG. 13. (Color online) Typical growth patterns in region r4. The patterns correspond to $t=90, 180$.

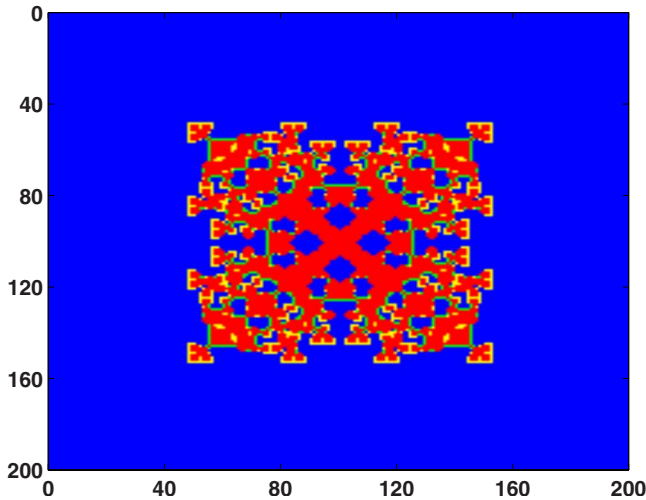


FIG. 18. (Color online) Typical growth patterns in region r_8 . The pattern corresponds to $t=50$.

r_2 (boundary 5) is $B+12A=3$; the boundary between region r_2 and region r_3 (boundary 1) is $3B+15A=10$; the boundary between region r_3 and region r_4 (boundary 6) is $6B+16A=21$; the boundary between region r_4 and region r_5 (boundary 7) is $6B+16A=28$; the boundary between region r_5 and region r_6 (boundary 8) is $3B+15A=28$ (see Table I with $p=0$).

Since different growth behaviors of D cluster and C cluster are identified in regions r_1 and r_2 , different cooperation behavior might occur in these two regions. We have studied cooperation frequency f in regions r_1 and r_2 . It is found $f \approx 0.3$ in region r_2 , which is very close to the cooperation frequency $f \approx 0.299$ in region $8/5 < b < 5/3$ in two-player PD. However, in region r_1 , simulations show cooperation frequency is notably larger: $f \approx 0.316$. This means the approximation calculation used in Ref. [7] (see Fig. 7 in Ref. [7]) becomes invalid.

Finally, we point out that on the boundaries, the symmetry will be broken. For example, let us consider parameters $B=2.8$, $A=0.7$, which is on the boundary $6B+16A=28$. Figure 16 shows the growth pattern of a 3×3 D cluster after two generations. For the sites with blue colored circled C , the maximum payoff is 28 and it can be achieved on sites with green colored C and sites with red colored circled D . As a result, the sites with blue colored circled C can be randomly

occupied by C or D , which breaks the symmetry. Note that the symmetry is also broken on boundaries in two-player PD.

B. $p=1/2$

In this subsection, we consider the case $p=1/2$. Figure 1(b) shows that the four regions remain qualitatively unchanged. In the special case $B=2A$, the three-player PD reduces to that corresponding to Eq. (2) [see the red line in Fig. 1(b)]. One can see that only three regions are observed along the red line, which is the same as in spatial two-player PD (without self-interaction).

Regions I and II still consist of four regions R1-R4 [see Fig. 4(c)]. The growth patterns in regions R1-R4 are the same as shown in Figs. 5–8. On the other hand, regions II and IV is still classified by five boundaries as in the case of $p=0$. However, due to variation of the boundaries, different number of regions are classified under different values of p . For example, Fig. 4(d) shows that eight regions exist when $p=1/2$. The growth patterns in regions r_1 - r_6 are the same as shown in Figs. 10–15. However, growth patterns in regions r_7 , r_8 are different, which are shown in Figs. 17 and 18.

In Fig. 4(c), one can see that only three growth patterns could be observed along the red line. In Fig. 4(d), one can see that only five growth patterns could be observed along the red line. This is the same as in spatial two-player PD (without self-interaction).

We study the cooperation frequency in regions r_1 , r_2 , r_7 , r_8 . It is found $f \approx 0.331$ in region r_1 , $f \approx 0.293$ in region r_2 , $f \approx 0.376$ in region r_7 , $f \approx 0.365$ in region r_8 . This further demonstrates that the approximation calculation used in Ref. [7] becomes invalid in spatial three-player PD.

C. General values of p

Table I gives the expressions of boundaries 1–8 at general values of p . Note that boundaries 2, 7, 8 are independent of p and boundaries 4 and 5 are identical. One can plot the regions of growth patterns of C cluster and D cluster from the expressions. We would like to mention that (i) when $p > p_c = 0.6773$ at which boundaries 1 and 2 intersect at $B=A$, region III will disappear [Fig. 1(c)]; (ii) with the increase of p , the growth patterns of D cluster also change: new region appears and some existing patterns gradually disappear (see, e.g., Fig. 19); (iii) with the increase of p , regions R3 and R2

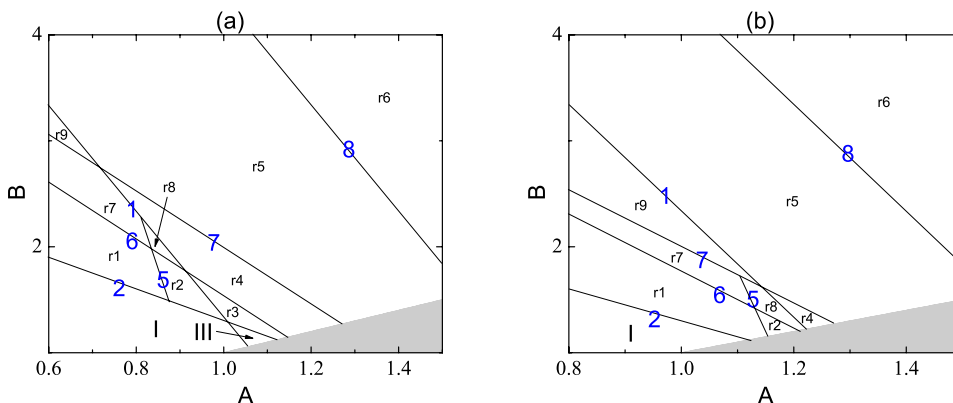


FIG. 19. (Color online) Regions of growth pattern of a 3×3 D cluster. (a) $p=0.6$, new region r_9 appears compared with Fig. 4(d), nevertheless, the growth pattern in region r_9 is the same as in region r_1 ; (b) $p=0.8$, region r_3 disappears compared with (a).

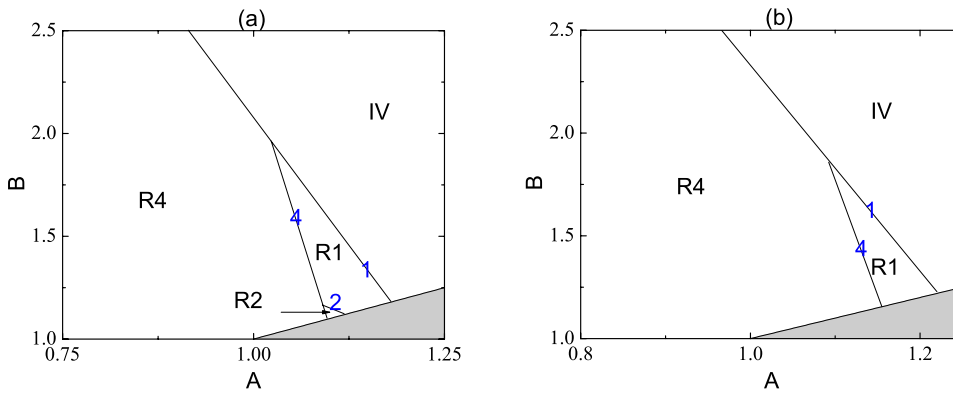


FIG. 20. (Color online) Regions of growth pattern of a 3×3 C cluster. (a) $p=0.75$, region R3 disappears; (b) $p=0.8$, region R2 disappears.

gradually disappear: region R3 disappears at $p=0.7414$ at which boundaries 3 and 4 intersect at $B=A$; region R2 disappears at $p=0.7707$ at which boundaries 2 and 4 intersect at $B=A$ (Fig. 20).

V. CONCLUSION

To summarize, we have extended traditional two-player PD to three-player PD. It is found four regions in the space are classified when $p < p_c = 0.6773$. While regions I, II, IV are similar to regions $b < 8/5$, $8/5 < b < 5/3$, $b > 5/3$ in two-player PD (without self-interaction), region III, in which neither a 3×3 cooperator cluster nor a 3×3 defector cluster could grow, has no counterpart in two-player PD. Nevertheless, when starting from a random distribution, it is found region III exhibits qualitatively similar pattern as in region IV. When $p > p_c$, region III disappears.

In addition, we find out there are more growth patterns of a 3×3 cooperator cluster and defector cluster than in two-player PD. Moreover, the number of growth patterns depends on p . In the special case of $p=1/2$ and $B=2A$, the three-player PD exhibits the same results as two-player PD. The analytical expressions for boundaries between regions I–IV and between different growth patterns are obtained.

Finally, multiple cooperation frequencies are identified in region II. This means the approximation calculation used in Ref. [7] becomes invalid.

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- [22] We would like to mention by rearranging the order of the parameters, the problem might correspond to other game. For example, it is reasonable to assume P is the smallest in a three-player snowdrift game.
- [23] Suppose $R < P$, then we have $T > r > P > R > p > S$. This means cooperators always score less than defectors. In this case, it is clear all players become defectors.